Math 436 (Spring 2020) - Homework 5

1. **Chapter 3:** 25

Solution: First, we show that the diagonal map $\Delta\colon X \to X \times X$ is continuous. Let p_1 and p_2 be the projection of $X \times X$ to the first and second copy of X respectively. Clearly, $p_1 \circ \Delta$ and $p_2 \circ \Delta$ are continuous. It follows from Theorem 3.13 that Δ is continuous.

For the second part, we have

X is Hausdorff

$\iff$ for any $x \neq y \in X$, there exist open neighborhoods U_x of x and U_y of y in X such that $U_x \cap U_y = \emptyset$.

$\iff$ for any $(x, y) \notin \Delta(X)$, there exists an open neighborhood $U_x \times U_y$ of (x, y) in $X \times X$ such that

$$(U_x \times U_y) \cap \Delta(X) = \emptyset.$$

$\iff$ $\Delta(X)$ is closed in $X \times X$.

2. Let $X \times Y$ be the product space of topological spaces X and Y. If $A \subseteq X$ and $B \subseteq Y$, prove that
   (a) $\overline{A \times B} = \overline{A} \times \overline{B}$;
   (b) $(A \times B)^{\circ} = \mathring{A} \times \mathring{B}$.

Solution:

(a) Since $\overline{A} \times \overline{B}$ is closed in $X \times Y$ and $A \times B \subseteq \overline{A} \times \overline{B}$, we have

$$\overline{A \times B} \subseteq \overline{A} \times \overline{B}.$$

On the other hand, if $x \in \overline{A}$ and $y \in \overline{B}$, then any open neighborhood U of x in X intersects A nonempty, i.e., $U \cap A \neq \emptyset$ and any open neighborhood V of y in Y intersects B nonempty, i.e., $V \cap B \neq \emptyset$. Note that any open neighborhood W of (x, y) in $X \times Y$ contains an open neighborhood of (x, y) of the form $U \times V$ where U is an open neighborhood of x in X and V is an open neighborhood of y in Y. Hence $W \cap (A \times B) \neq \emptyset$. This shows that $(x, y) \in \overline{A \times B}$.

(b) Omitted.

3. If X and Y are discrete spaces, then the product space $X \times Y$ is discrete.

Solution: A discrete space is a topological space where every single point is open subset. Let (x, y) be a point in $X \times Y$. $\{x\}$ is open in X and $\{y\}$ is open in Y, since X and Y are discrete spaces. It follows that $\{(x, y)\} = \{x\} \times \{y\}$ is open in $X \times Y$. This shows that $X \times Y$ is discrete.

4. If X and Y are indiscrete spaces, then the product space $X \times Y$ is indiscrete.

Solution: The only open subsets of an indiscrete space are $\emptyset$ and the whole space. It follows that the only open subsets of the product topology on $X \times Y$ are $\emptyset \times \emptyset = \emptyset$, $\emptyset \times Y = \emptyset$, $X \times \emptyset = \emptyset$ and $X \times Y$. This shows that $X \times Y$ is indiscrete.

5. Let (X, d_X) and (Y, d_Y) be two metric spaces. Consider the formula

$$D((x_1, y_1), (x_2, y_2)) \coloneqq \sqrt{d_X(x_1, x_2)^2 + d_Y(y_1, y_2)^2}$$

for all (x_1, y_1) and (x_2, y_2) in $X \times Y$. Prove that D defines a metric on the set $X \times Y$.

Solution:

(1) Clearly, $D((x_1, y_1), (x_2, y_2)) \geq 0$ and

$$D((x_1, y_1), (x_2, y_2)) = 0 \iff (x_1, y_1) = (x_2, y_2).$$

(2) Also, it is obvious that

$$D((x_1, y_1), (x_2, y_2)) = D((x_2, y_2), (x_1, y_1)).$$

(3) Now we want to show that

$$D((x_1, y_1), (x_3, y_3)) \leq D((x_1, y_1), (x_2, y_2)) + D((x_2, y_2), (x_3, y_3)).$$

Since d_X and d_Y are metrics on X and Y respectively, we have

$$d_X(x_1, x_3) \leq d_X(x_1, x_2) + d_X(x_2, x_3)$$

and

$$d_Y(y_1, y_3) \leq d_Y(y_1, y_2) + d_Y(y_2, y_3).$$

By squaring both sides, we have

$$d_X(x_1, x_3)^2 \leq d_X(x_1, x_2)^2 + 2d_X(x_1, x_2)d_X(x_2, x_3) + d_X(x_2, x_3)^2$$
$$d_Y(y_1, y_3)^2 \leq d_Y(y_1, y_2)^2 + 2d_Y(y_1, y_2)d_Y(y_2, y_3) + d_Y(y_2, y_3)^2.$$

It follows that

$$
\begin{aligned}
&D((x_1, y_1), (x_3, y_3))^2 \\
&= d_X(x_1, x_3)^2 + d_Y(y_1, y_3)^2 \\
&\leq d_X(x_1, x_2)^2 + 2d_X(x_1, x_2)d_X(x_2, x_3) + d_X(x_2, x_3)^2 \\
&\quad + d_Y(y_1, y_2)^2 + 2d_Y(y_1, y_2)d_Y(y_2, y_3) + d_Y(y_2, y_3)^2 \\
&\leq d_X(x_1, x_2)^2 + d_Y(y_1, y_2)^2 \\
&\quad + 2 \cdot \sqrt{d_X(x_1, x_2)^2 + d_Y(y_1, y_2)^2} \cdot \sqrt{d_X(x_2, x_3)^2 + d_Y(y_2, y_3)^2} \\
&\quad + d_X(x_2, x_3)^2 + d_Y(y_2, y_3)^2 \\
&= \big(D((x_1, y_1), (x_2, y_2)) + D((x_2, y_2), (x_3, y_3))\big)^2
\end{aligned}
$$

where we have used Cauchy-Schwartz inequality

$$
\begin{aligned}
&d_X(x_1, x_2)d_X(x_2, x_3) + d_Y(y_1, y_2)d_Y(y_2, y_3) \\
&\leq \sqrt{d_X(x_1, x_2)^2 + d_Y(y_1, y_2)^2}\sqrt{d_X(x_2, x_3)^2 + d_Y(y_2, y_3)^2}.
\end{aligned}
$$

Recall that for vectors $\mathbf{v} = (d_X(x_1, x_2), d_Y(y_1, y_2))$ and $\mathbf{w} = (d_Y(y_1, y_2), d_Y(y_2, y_3))$ in $\mathbb{R}^2$, we have

$$\mathbf{v} \cdot \mathbf{w} \leq \|\mathbf{v}\|\|\mathbf{w}\|.$$

This finishes the proof.

6. (Bonus Question) Show that the topology on $X \times Y$ determined by the metric D given in Question #5 is exactly the product topology on $X \times Y$.

Solution: We make the following observations.

(1) The family of metric balls $\{B_r(x) \mid \forall x \in X, r > 0\}$ is a basis for the topology of (X, d_X).

(2) The family of metric balls $\{B_s(y) \mid \forall y \in Y, s > 0\}$ is a basis for the topology of (Y, d_Y).

(3) The family of metric balls $\beta = \{B_t((x, y)) \mid \forall (x, y) \in X \times Y, t > 0\}$ is a basis for the topology of $(X \times Y, D)$.

(4) The family $\alpha = \{B_r(x) \times B_s(y) \mid \forall (x, y) \in X \times Y, r > 0 \text{ and } s > 0\}$ is a basis for the product topology on $X \times Y$.

To prove the statement, it suffices to show that every member of β is a union of members of α and every member of α is a union of members of β.

Given a member $B_t((x, y))$ of β, let (a, b) be any point in $B_t((x, y))$. We want to show that there exist $r > 0$ and $s > 0$ such that $B_r(a) \times B_s(b) \subseteq B_t((x, y))$. Since

$(a,b) \in B_t((x,y))$, we have

$$D((a,b),(x,y)) = \sqrt{d_X(a,x)^2 + d_Y(b,y)^2} < t.$$

In particular, there exists $\varepsilon > 0$ such that

$$\sqrt{d_X(a,x)^2 + d_Y(b,y)^2} = t - \varepsilon.$$

Let $r = s = \varepsilon/2$. Then for any $(w,z) \in B_r(a) \times B_s(b)$, we have

$$d_X(a,w) < \varepsilon/2 \text{ and } d_Y(b,z) < \varepsilon/2.$$

which implies that

$$D((w,z),(a,b)) = \sqrt{d_X(w,a)^2 + d_Y(z,b)^2} < \frac{\sqrt{2}}{2}\varepsilon < \varepsilon$$

It follows that

$$\begin{aligned} D((w,z),(x,y)) &\leq D((w,z),(a,b)) + D((a,b),(x,y)) \\ &< \varepsilon + t - \varepsilon \\ &= t \end{aligned}$$

Therefore, $B_{\varepsilon/2}(a) \times B_{\varepsilon/2}(b) \subseteq B_t((x,y))$. So we have shown that $B_t((x,y))$ is a union of members of α.

The proof for showing that every member of α is a union of members of β is similar. I will omit this part of the proof.